Note: A small subset of essential problems are marked with a red star (★). We especially encourage you to try these out before recitation.

Problem 1. ★

Consider the truth table on the right, which defines two functions F and G of three input variables (A, B, and C).

For each function, write it in normal form, then find a minimal sum of products (minimal SOP) expression.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>

Normal form for F(A,B,C) = \( \overline{A} \cdot B \cdot C + \overline{A} \cdot B \cdot \overline{C} + \overline{A} \cdot B \cdot \overline{C} + A \cdot \overline{B} \cdot C + A \cdot B \cdot C \)

Minimal sum of products for F(A,B,C) = \( \overline{A} \cdot \overline{B} + B \cdot C + \overline{B} \cdot \overline{C} \)

Normal form for G(A,B,C) = \( \overline{A} \cdot B \cdot C + \overline{A} \cdot B \cdot C + \overline{A} \cdot B \cdot \overline{C} + A \cdot \overline{B} \cdot \overline{C} + A \cdot B \cdot \overline{C} \)

\[
\begin{align*}
&= \overline{A} \cdot \overline{B} + \overline{A} \cdot C + A \cdot \overline{C} \\
&= \overline{A} \cdot B + \overline{C} \cdot (A + \overline{A})
\end{align*}
\]

Minimal sum of products for G(A,B,C) = \( \overline{A} \cdot \overline{B} + \overline{C} \)
Problem 2. ★

Consider the 3-input Boolean function \( G(A,B,C) = \overline{A} \overline{C} + A \overline{B} + \overline{B} \overline{C} \).

1. How many 1’s are there in the output column of G’s 8-row truth table?
   4

2. Give a minimal sum-of-products expression for G.
   \( \overline{A} \cdot \overline{C} + A \cdot B \)

3. There’s good news and bad news: the bad news is that the stockroom only has G gates. The good news is that it has as many as you need. Using only combinational circuits built from G gates, one can implement (choose the best response):
   (A) Any Boolean function (G is functionally complete)
   (B) Only functions with 3 inputs or less
   (C) Only functions with the same truth table as G

   \[
   G(A,B,0) = \overline{A} + A \cdot \overline{B} + \overline{B} \\
   = \overline{A} + \overline{B} \cdot (A + 1) \\
   = \overline{A} + \overline{B} \\
   = A \cdot \overline{B}
   \]

   NAND is functionally complete.

4. Can a sum-of-products expression involving 3 input variables with greater than 4 product terms always be simplified to a sum-of-products expression using fewer product terms?

   Yes. There will always be at least two rows that differ in exactly one bit and those rows can be merged into one product term.
**Problem 3. ★**

Consider the logic diagram shown below, which includes XOR2, NAND2, NAND3, and INV (inverter) gates.

![Logic Diagram](image)

<table>
<thead>
<tr>
<th>Gate</th>
<th>( t_{PD} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>INV</td>
<td>1.0ns</td>
</tr>
<tr>
<td>NAND2</td>
<td>1.5ns</td>
</tr>
<tr>
<td>NAND3</td>
<td>1.8ns</td>
</tr>
<tr>
<td>XOR2</td>
<td>2.5ns</td>
</tr>
</tbody>
</table>

1. Using the \( t_{PD} \) information for the gate components shown in the table above, compute the \( t_{PD} \) for the circuit.

\[
t_{PD,D} = 2 \times t_{PD,XOR} = 5\text{ns}
\]

\[
t_{PD,Bout} = t_{PD,INV} + t_{PD,NAND2} + t_{PD,NAND3} = 1 + 1.5 + 1.8 = 4.3\text{ns}
\]

\[
t_{PD} = 5\text{ns}
\]

2. Find minimal sum-of-products expressions for both outputs, \( D \) and \( Bout \).

NOTE: The gates implement the following functions:
- \( NAND2(a,b) = \overline{a \cdot b} \)
- \( NAND3(a,b,c) = \overline{a \cdot b \cdot c} \)
- \( XOR2(a,b) = a \cdot \overline{b} + \overline{a} \cdot b \)

\[
D(X,Y,\text{Bin}) = (X \cdot \overline{Y} + \overline{X} \cdot Y) \cdot \overline{B_{in}} + (X \cdot \overline{Y} + \overline{X} \cdot Y) \cdot B_{in}
\]

\[
= X \cdot \overline{Y} \cdot \overline{B_{in}} + \overline{X} \cdot Y \cdot \overline{B_{in}} + X \cdot Y \cdot B_{in} + \overline{X} \cdot \overline{Y} \cdot B_{in}
\]

\[
B_{out}(X,Y,\text{Bin}) = \overline{B_{in}} \cdot Y \cdot \overline{B_{in}} \cdot \overline{X} \cdot \overline{Y} = B_{in} \cdot Y + B_{in} \cdot \overline{X} + Y \cdot \overline{X}
\]

Minimal sum of products for \( D(X,Y,\text{Bin}) = X \cdot \overline{Y} \cdot \overline{B_{in}} + \overline{X} \cdot Y \cdot \overline{B_{in}} + X \cdot Y \cdot B_{in} + \overline{X} \cdot \overline{Y} \cdot B_{in} \)

Minimal sum of products for \( Bout(X,Y,\text{Bin}) = B_{in} \cdot Y + B_{in} \cdot \overline{X} + Y \cdot \overline{X} \)
Problem 4.

Simplify the following Boolean expressions by finding a minimal sum-of-products expression for each one:

1. $ac + b + c$
2. $(a + b)c + \bar{c}a + b(\bar{a} + c)$
3. $a(b + \bar{c})(b + a(b + c))$
4. $a(b + c(d + ef))$

1. $ac + b + c = b + c = b \cdot \bar{c}$

2. $(a + b)c + \bar{c}a + b(\bar{a} + c) = ac + bc + \bar{c}a + b\bar{a} = a + bc + \bar{a}b = a + b$

3. $a(b + \bar{c})(b + a(b + c)) = a \cdot \bar{b} \cdot \bar{c} \cdot (b + ab + ac) = a \cdot \bar{b} \cdot \bar{c} \cdot (b + ac) = abbc + abc\bar{c} = 0$

4. $a(b + c(d + ef)) = a(b + cd + cef) = ab + acd + acef$
Problem 5.

There are some Boolean expressions for which no assignment of values to variables can produce True (e.g., \( a\bar{a} \)). Those Boolean expressions are said to be non-satisfiable. Are the following Boolean expressions satisfiable? If the expression is satisfiable, give an assignment to variables that makes the expression evaluate to True. If the expression is non-satisfiable, prove it.

1. \((a + b)c + \bar{c}a + b(\bar{a} + c)\)
2. \((x + y)(x + \bar{y})(z + \bar{y})(y + \bar{x})\)
3. \((x + y + z)(x + y + \bar{z})(x + y + \bar{z})(\bar{x} + y + z) \cdot (x + \bar{y} + \bar{z})(\bar{x} + y + z)(\bar{x} + \bar{y} + \bar{z})\)
4. \(xyz + xy\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}y\bar{z}\)

1. Satisfiable. Possible solution:
   \( a=1, b=1, c=1 \)

2. Satisfiable. Possible solution:
   \( x=1, y=1, z=1 \)

3. Non-satisfiable. All 8 combinations of \( x+y+z \) are present, so it is impossible to satisfy them all. We can also prove this by reducing the expression:
   \((x + y + z)(x + y + \bar{z})(x + y + z)(\bar{x} + y + z)(\bar{x} + \bar{y} + \bar{z})(\bar{x} + \bar{y} + z)(\bar{x} + \bar{y} + \bar{z})\)
   \(= (x + y)(x + \bar{y})(\bar{x} + y)(\bar{x} + y) = xx = 0 \)

4. Satisfiable. Possible solution:
   \( x=0, y=0, z=1 \)
Problem 6. (Quiz 1 Spring 2018, 12/100 points)

(A) (6 points) Simplify the following Boolean expressions by finding a minimal sum-of-products expression for each one. (Note: These expressions can be reduced into a minimal SOP by repeatedly applying the Boolean algebra properties we saw in lecture.)

1. \((a + b \cdot \bar{c}) \cdot d + c = \bar{a} \cdot \bar{b} \cdot d + c\)

2. \(a \cdot (b + c) (c + a) = a \cdot \bar{b} \cdot \bar{c}\)

(B) (6 points) There are Boolean expressions for which no assignment of values to variables can produce True (e.g., \(a \cdot \bar{a}\)). These Boolean expressions are said to be non-satisfiable.

Are the following Boolean expressions satisfiable? If the expression is satisfiable, give an assignment to variables that makes the expression evaluate to True. If the expression is non-satisfiable, explain why.

1. \((\bar{x} + y\bar{z}) \cdot (\bar{y}x + z) \cdot (\bar{z}y + x)\)
   \[\bar{xz}\]
   \[\bar{xz}\bar{y} + \bar{xz}x = 0 \rightarrow \text{Unsatisfiable}\]

2. \((\bar{x} + y\bar{z}) \cdot (\bar{y}x + z) \cdot (\bar{z}y + x) + (\bar{x} + yz) \cdot (\bar{y}x + z) \cdot (\bar{z}y + x)\)
   \[0 \text{ (same as above)}\]
   \[xyz\]
   Satisfiable with \(x=1 \ y=1 \ z=1\)