6.004 Tutorial Problems
L08 – Combinational Logic 2

Note: A small subset of essential problems are marked with a red star (★). We especially encourage you to try these out before recitation.

Problem 1. ★

We want to implement a parametric function reverse#(n) that reverses the bits of its n-bit input argument. For example, if Bit#(4) x = {a, b, c, d}, then reverse#(4)(x) should return {d, c, b, a}.

(A) Implement reverse#(n) by recursing on the parameter n (i.e., calling reverse#(k) with k < n). You cannot use a for loop.

```plaintext
function Bit#(n) reverse#(Integer n)(Bit#(n) x);

return (n == 1)? x : {x[0], reverse#(n-1)(x[n-1:1])};

endfunction
```

(B) Implement reverse#(n) using a for loop.

```plaintext
function Bit#(n) reverse#(Integer n)(Bit#(n) x);

Bit#(n) res = 0;
for (Integer i = 0; i < n; i = i + 1)
    res[i] = x[n-1-i];
return res;

endfunction
```
Problem 2.

Parameterize the bit-scan-reverse function from Lab 3 to take as input an n-bit vector and output the index of the first non-zero bit scanned from the largest index (i.e., the position of the most-significant 1). **Assume that the parameter n is a power of 2 and n >= 2.**

(A) Implement \( \text{bitScanReverse}(n) \) without using a for loop.

```plaintext
function Bit#(log2(n)) bitScanReverse#(Integer n)(Bit#(n) x);

if (n == 2) return x[1];
else begin
  let upper = bitScanReverse#(n/2)(x[n-1:n/2]);
  let lower = bitScanReverse#(n/2)(x[n/2-1:0]);
  return (upper == 0)? {1'b0, lower} : {1'b1, upper};
end
endfunction
```

NOTE: You could also avoid the if-else statement by defining the base case function \( \text{bitScanReverse}(1) = x[1] \) separately.

(B) Implement \( \text{bitScanReverse}(n) \) using a for loop.

```plaintext
function Bit#(log2(n)) bitScanReverse#(Integer n)(Bit#(n) x);

Bit#(log2(n)) res = 0;
for (Integer i = 0; i < n; i = i + 1)
  res = (in[i] == 1) ? i : res;
return res;
endfunction
```

(C) When synthesized manually (i.e., without logic optimizations, just the gates that your circuit expresses), how does propagation delay grow with the number of input bits for each implementation? Use order-of notation.

Without a loop (A), delay grows with \( \Theta(\log n) \), as the implementation uses a balanced tree of \( \text{bitScanReverse}() \) functions (at each step, each additional function processes half of the bits, and the upper and lower halves are computed in parallel).

With a loop (B), delay grows with \( \Theta(n) \), as the implementation uses a chain of muxes on res.

(Different answers are possible, depending on your implementations.)
Problem 3.

In Lab 3, we wrote a function \texttt{isPowerOfTwo} that computes whether a 4-bit input was a power of 2 or not. This function checks whether there is only one bit in the input that is equal to 1.

We want to generalize \texttt{isPowerOfTwo} by rewriting it as a parametric function that works with inputs of arbitrary bit-width.

(A) Implement \texttt{isPowerOfTwo#(\texttt{n})} using a \texttt{for} loop.

```vhdl
function Bool isPowerOfTwo#(Integer n)(Bit#(n) x);
    Bool someOnes = False;
    Bool twoOrMoreOnes = False;
    for (Integer i = 0; i < n; i = i + 1)
        if (x[i] == 1) begin
            twoOrMoreOnes = someOnes;
            someOnes = True;
        end
    return someOnes && !twoOrMoreOnes;
endfunction
```

(B) If your implementation above has $\Theta(n)$ propagation delay when synthesized manually (i.e., without logic optimizations, just the gates that your circuit expresses), then rewrite \texttt{isPowerOfTwo#(\texttt{n})} so that it has $\Theta(\log n)$ propagation delay.

\textit{Hint:} Since you used a \texttt{for} loop above, you probably have a linear chain of gates in your design. Instead, think about how to solve this problem by composing functions so that, at each step, you halve the number of input bits each function processes. This will produce a tree of gates with logarithmic depth. You’ll likely need to use an auxiliary parametric function that recurses on its own parameter.

```vhdl
typedef enum { ZeroOnes, OneOne, TwoOrMoreOnes } Pow2Res;

function Pow2Res pow2#(1)(Bit#(n) x);
    return (x == 1)? OneOne : ZeroOnes;
endfunction

function Pow2Res pow2#(Integer n)(Bit#(n) x);
    let lower = pow2#(n/2)(x[n/2-1:0]);
    let upper = pow2#(n-n/2)(x[n-1:n/2]);
    return (lower == ZeroOnes && upper == ZeroOnes)? ZeroOnes :
        ((lower == OneOne && upper == ZeroOnes) ||
         (lower == ZeroOnes && upper == OneOne))? OneOne :
            TwoOrMoreOnes;
endfunction

function Bool isPowerOfTwo#(Integer n)(Bit#(n) x);
    return pow2#(n)(x) == OneOne;
endfunction
```
Problem 4. (adapted from Quiz 1 Fall 2018, 20/100 points)
(NOTE: Part C of this question will be much easier once you have completed Lab 4)

(A) (5 points) The following parametric function $f$ performs a basic operation using $a$ and $b$. We want $f_2$ to implement the same function as $f$. Fill in the blank in $f_2$ to make the two functions equivalent. Write a single-line expression that uses the ternary operator (? :).

```
function Bit#(n) f(Integer n)(Bit#(n) a, Bit#(1) b);
    Bit#(n) x = 0;
    for (Integer i = 0 ; i < n ; i = i+1) begin
        x[i] = a[i] ^ b;
    end
    return x;
endfunction

function Bit#(n) f2(Integer n)(Bit#(n) a, Bit#(1) b);
    return (b==1) ? ~a : a;
endfunction
```

(B) (5 points) Write the truth table for the combinational device described by the function below.

```
function Bit#(2) f(Bit#(1) a, Bit#(1) b, Bit#(1) c);
    Bit#(2) ret = zeroExtend(a) + signExtend(b);
    case ({a,b})
        0: ret = {1, c};
        2: ret = {a ^ b, a & b};
        3: ret = ~signExtend(c);
    endcase
    return ret;
endfunction
```

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<tr>
<th>a</th>
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(C) (5 points) The following parametric function \( g \) performs a specific arithmetic operation on \( n \)-bit operands \( a \) and \( b \). We want the function \( g_2 \) to implement \( g \) in a single line of code. Fill in the blank with a single expression to make \( g_2 \) equivalent to \( g \).

```markdown
function Bit#(1) g#(Integer n)(Bit#(n) a, Bit#(n) b);
    Bit#(2) ret = 'b10;
    for (Integer i = n-1 ; i >= 0 ; i = i-1) begin
        if ({a[i], b[i]} == 'b01) ret = {0, ret[1] | ret[0]};
        else if ({a[i], b[i]} == 'b10) ret = {0, ret[0]};
    end
    return ret[1] | ret[0];
endfunction

function Bit#(1) g2#(Integer n)(Bit#(n) a, Bit#(n) b);
    return (a <= b)? 1 : 0_______________________;
endfunction
```

(D) (5 points) Finish the following circuit diagram to implement function `computeB`, given below. You may only use 32-bit 2-to-1 multiplexers, constants (0, 1, 2, 3, ...) and logic gates (AND, NOT, OR, XOR). We have provided three 32-bit greater-than-or-equal (\(>=\)) comparators for you.

```markdown
function Bit#(32) computeB(Bit#(32) in);
    Bit#(32) out = 0;
    if ( in >= 1 ) out = 1;
    if ( in >= 5 ) out = 5;
    if ( in >= 10 ) out = 10;
    return out;
endfunction
```

![Circuit Diagram](image)